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One-Parameter Homogeneous Differential Realization and Boson–Fermion Realization of the SPL(2,1) Superalgebra

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One-parameter homogeneous differential realization of the SPL(2,1) superalgebra on the space of homogeneous polynomials and the corresponding boson–fermion realization are studied. The parameter α may be related to the interaction parameter U in one exactly solvable model for correlated electrons.

KEY WORDS: SPL(2,1) superalgebra; homogeneous differential realization; boson–fermion realization; exactly solvable model.

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1. INTRODUCTION

Lie superalgebras have played an important role in nuclear physics, superunification, and in supergravity (Balantekin and Bars, 1982). A series of models of correlated electrons on a lattice and exactly solvable in one dimension and supersymmetric, such as Hubbard and extended Hubbard models and *t–J* model, EKS model, BGLZ model (Brachen *et al.*, 1995), has been extensively studied due to their promising role in theoretical condensed-matter physics and possibly in high-T_c superconductivity. Those models contain one symmetry-preserving free real parameter which is the Hubbard interaction parameter U. Quasi-exactly solvable problems (QESP) in quantum mechanics have been discussed by Turbiner and Ushveridze (1987). QESP in quantum mechanics have become increasingly important because they have been generalized to the study of the conformal field theory. A connection of QESP and finite-dimensional inhomogeneous differential realizations of Lie algebras (or superalgebras) has been described at the first time by Turbiner (Dirac, 1984; Shifman and Turbiner, 1989; Turbiner, 1988, 1992).

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The key resolving the QESP lies in studying finite-dimensional inhomogeneous differential realizations of Lie (super)algebras. Some homogeneous and inhomogeneous differential realizations of Lie superalgebras SPL(2,1) and GL(2|1) have been given by Chen (1993, 2000, 2001a,b). Therefore, it is very important to study further the new one-parameter homogeneous and inhomogeneous differential realizations of the SPL(2,1) superalgebra. In the present paper we shall be concerned with the SPL(2,1) superalgebra. The purpose of the present paper is to obtain one-parameter homogeneous differential realization of the SPL(2,1) superalgebra and the corresponding boson–fermion realization. The details of one-parameter inhomogeneous differential realizations are deferred to separate publication. This paper is arranged as follows. In Section 2 by introducing a typical four-dimensional one parameter elementary representation we derive homogeneous differential realization of the SPL(2,1) on the spaces of homogeneous polynomials. In Section 3 we consider their corresponding relations of C-number differential operators and boson creation and annihilation operators, of Grassmann number differential operators and fermion creation and annihilation operators respectively. The corresponding boson–fermion realization of the SPL(2,1) superalgebra is obtained in terms of the homogeneous differential realization.

2. ONE-PARAMETER HOMOGENEOUS DIFFERENTIAL REALIZATION OF THE SPL(2,1)

In accordance with Chen (1993) the generators of the SPL(2,1) superalgebra read as follows:

$$\{Q_3, Q_+, Q_-, B \in \mathrm{SPL}(2,1)_{\bar{0}} \,|\, V_+ , V_-, W_+, W_- \in \mathrm{SPL}(2,1)_{\bar{1}}\} \tag{1}$$

and satisfy the following commutation and anticommutation relations:

$$\begin{aligned}
&[Q_3, Q_\pm] = \pm Q_\pm, \quad [Q_+, Q_-] = 2Q_3, \quad [B, Q_\pm] = [B, Q_3] = 0 \\
&[Q_3, V_\pm] = \pm\frac{1}{2}V_\pm, \quad [Q_3, W_\pm] = \pm\frac{1}{2}W_\pm, \quad [B, V_\pm,] = \frac{1}{2}V_\pm \\
&[B, W_\pm] = -\frac{1}{2}W_\pm, \quad [Q_\pm, V_\mp] = V_\pm, \quad [Q_\pm, W_\mp] = W_\pm \quad [Q_\pm, V_\pm] = 0 \\
&[Q_\pm, W_\pm] = 0, \quad \{V_\pm, V_\pm\} = \{V_\pm, V_\mp\} = \{W_\pm, W_\pm\} = \{W_\pm, W_\mp\} = 0, \\
&\{V_\pm, W_\pm\} = \pm Q_\pm, \quad \{V_\pm, W_\mp\} = -Q_3 \pm B.
\end{aligned} \tag{2}$$

We consider a typical four-dimensional one parameter elementary representation.

$$D(Q_3) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad D(Q_+) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3}$$

$$D(Q_-) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad D(B) = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & 1/2+\alpha & 0 & 0 \\ 0 & 0 & 1/2+\alpha & 0 \\ 0 & 0 & 0 & 1+\alpha \end{bmatrix},$$

From (2) and (3), we can obtain

$$D(V_+) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{\alpha} & 0 & 0 & 0 \\ 0 & \sqrt{\alpha+1} & 0 & 0 \end{bmatrix}, \quad D(V_-) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{\alpha} & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{\alpha+1} & 0 \end{bmatrix},$$

$$D(W_+) = \begin{bmatrix} 0 & 0 & -\sqrt{\alpha} & 0 \\ 0 & 0 & 0 & \sqrt{\alpha+1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad D(W_-) = \begin{bmatrix} 0 & \sqrt{\alpha} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\alpha+1} \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4}$$

with real parameter $\alpha > 0$. In order to study differential realization of the SPL(2,1) superalgebra on the space of homogeneous polynomials, introducing four independent variables $\mu_1, \mu_2, \xi_1, \xi_2$ where μ_1, μ_2 are C-numbers and ξ_1, ξ_2 are Grassmann numbers respectively, we regard them as the basis of representation space, i.e.,

$$\mu_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \mu_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \xi_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \xi_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{5}$$

Noting (3–5), we have

$$Q_3\mu_1 = \frac{1}{2}\mu_1 \quad Q_3\mu_2 = -\frac{1}{2}\mu_2, \quad Q_3\xi_1 = 0, \quad Q_3\xi_2 = 0,$$

$$Q_+\mu_1 = 0, \quad Q_+\mu_2 = \mu_1, \quad Q_+\xi_1 = 0, \quad Q_+\xi_2 = 0,$$

$$Q_-\mu_1 = \mu_2, \quad Q_-\mu_2 = 0, \quad Q_-\xi_1 = 0, \quad Q_-\xi_2 = 0,$$

$$B\mu_1 = \left(\frac{1}{2}+\alpha\right)\mu_1, \quad B\mu_2 = \left(\frac{1}{2}+\alpha\right)\mu_2, \quad B\xi_1 = \alpha\xi_1, \quad B\xi_2 = (1+\alpha)\xi_2,$$

$$V_+\mu_1 = 0, \quad V_+\mu_2 = \sqrt{1+\alpha}\xi_2, \quad V_+\xi_1 = \sqrt{\alpha}\mu_1, \quad V_+\xi_2 = 0, \tag{6}$$

$$V_-\mu_1 = -\sqrt{1+\alpha}\xi_2, \quad V_-\mu_2 = 0, \quad V_-\xi_1 = \sqrt{\alpha}\mu_2, \quad V_-\xi_2 = 0,$$
$$W_+\mu_1 = 0, \quad W_+\mu_2 = -\sqrt{\alpha}\xi_1, \quad W_+\xi_1 = 0, \quad W_+\xi_2 = \sqrt{1+\alpha}\mu_1,$$
$$W_-\mu_1 = \sqrt{\alpha}\xi_1, \quad W_-\mu_2 = 0, \quad W_-\xi_1 = 0, \quad W_-\xi_2 = \sqrt{1+\alpha}\mu_2.$$

Using differential operators the generators of the SPL(2,1) are constructed as follows:

$$\begin{aligned}
Q_3 &= \frac{1}{2}\left(\mu_1\frac{\partial}{\partial\mu_1} - \mu_2\frac{\partial}{\partial\mu_2}\right), \quad Q_+ = \mu_1\frac{\partial}{\partial\mu_2}, \quad Q_- = \mu_2\frac{\partial}{\partial\mu_1}\\
B &= \left(\frac{1}{2}+\alpha\right)\left(\mu_1\frac{\partial}{\partial\mu_1} + \mu_2\frac{\partial}{\partial\mu_2}\right) + \alpha\left(\xi_1\frac{\partial}{\partial\xi_1} + \xi_2\frac{\partial}{\partial\xi_2}\right)\\
V_+ &= \sqrt{\alpha}\mu_1\frac{\partial}{\partial\xi_1} + \sqrt{1+\alpha}\xi_2\frac{\partial}{\partial\mu_2},\\
V_- &= \sqrt{\alpha}\mu_2\frac{\partial}{\partial\xi_1} - \sqrt{1+\alpha}\xi_2\frac{\partial}{\partial\mu_1},\\
W_+ &= -\sqrt{\alpha}\xi_1\frac{\partial}{\partial\mu_2} + \sqrt{1+\alpha}\mu_1\frac{\partial}{\partial\xi_2},\\
W_- &= \sqrt{\alpha}\xi_1\frac{\partial}{\partial\mu_1} + \sqrt{1+\alpha}\mu_2\frac{\partial}{\partial\xi_2}.
\end{aligned} \tag{7}$$

It is easily proved that the generators thus represented satisfy all the commutation and anticommutation relations of the SPL(2,1). Substantially, Eq. (7) is a differential realization on the space of homogeneous polynomials of degree one, i.e., $A_1 = \{\mu_1, \mu_2, \xi_1, \xi_2\}$. For the space of homogeneous polynomials of degree n

$$A_n = \left\{\mu_1^{i_1}\mu_2^{i_2}\xi_1^{k_1}\xi_2^{k_2}\,|i_1\,, i_2 \in Z^+, k_1, k_2 = 0, 1, i_1+i_2+k_1+k_2 = n\right\} \tag{8}$$

where Z^+denotes the set of all non-negative integer, it carries the direct product representation of the SPL(2,1),

$$D_s^{\otimes n} = \underbrace{(D\otimes D\otimes\cdots\otimes D)}_{\text{deg reen}}\text{ symmetrized} \tag{9}$$

Using the defination of direct product representation,

$$\begin{aligned}
\hat{F}\left(\mu_1^{i_1}\mu_2^{i_2}\xi_1^{k_1}\xi_2^{k_2}\right) &= \left(F\mu_1^{i_1}\right)\mu_2^{i_2}\xi_1^{k_1}\xi_2^{k_2} + \mu_1^{i_1}\left(F\mu_2^{i_2}\right)\xi_1^{k_2}\xi_2^{k_2} + \mu_1^{i_1}\mu_2^{i_2}\left(F\xi_1^{k1}\right)\xi_2^{k_2}\\
&\quad + \mu_1^{i_1}\mu_2^{i_2}\xi_1^{k_1}\left(F\xi_2^{k_2}\right),
\end{aligned} \tag{10}$$

where F stands for any generator of the SPL(2,1), we can obtain its differential realization F on A_n. It is easy to check that $\hat{F} = F$.

3. ONE-PARAMETER BOSON–FERMION REALIZATION OF THE SPL(2,1)

Considering their corresponding relations of C-number differential operators $(\mu_i, \frac{\partial}{\partial\mu_i})$ and boson creation and annihilation operators (b_i^+, b_i),

$$b_i^+ \Leftrightarrow \mu_i, \quad b_i \Leftrightarrow \frac{\partial}{\partial\mu_i} \quad [b_i, b_j^+] = \delta_{ij}, \quad \left[\frac{\partial}{\partial\mu_i}, \mu_j\right] = \delta_{ij} \tag{11}$$
$$[b_i, b_j] = [b_i^+, b_j^+] = 0, \quad \left[\frac{\partial}{\partial\mu_i}, \frac{\partial}{\partial\mu_j}\right] = [\mu_i, \mu_j] = 0$$

and of Grassmann number differential operators $(\xi_i, \frac{\partial}{\partial\xi_i})$ and fermion creation and annihilation operators (a_i^+, a_i), respectively,

$$a_i^+ \Leftrightarrow \xi_i, \quad a_i \Leftrightarrow \frac{\partial}{\partial\xi_i}, \quad \{a_i, a_j^+\} = \delta_{ij}, \quad \left\{\frac{\partial}{\partial\xi_i}, \frac{\partial}{\partial\xi_j}\right\} = \delta_{ij} \tag{12}$$
$$\{a_i, a_j\} = \{a_i^+, a_j^+\} = 0, \left\{\frac{\partial}{\partial\xi_i}, \frac{\partial}{\partial\xi_j}\right\} = \{\xi_i, \xi_j\} = 0$$

the corresponding homogeneous boson–fermion realization of the SPL(2,1) is obtained in terms of two pairs of boson operators and two pairs of fermion operators as follows:

$$\begin{aligned}
Q_3 &= \frac{1}{2}(b_1^+ b_1 - b_2^+ b_2), \quad Q_+ = b_1^+ b_2, \quad Q_- = b_2^+ b_1, \\
B &= \left(\frac{1}{2} + \alpha\right)(b_1^+ b_1 + b_2^+ b_2) + \alpha(a_1^+ a_1 + a_2^+ a_2), \\
V_+ &= \sqrt{\alpha} b_1^+ a_1 + \sqrt{1+\alpha} a_2^+ b_2, \quad V_- = \sqrt{\alpha} b_2^+ a_1 - \sqrt{1+\alpha} a_2^+ b_1 \\
W_+ &= -\sqrt{\alpha} a_1^+ b_2 + \sqrt{1+\alpha} b_1^+ a_2, \quad W_- = \sqrt{\alpha} a_1^+ b_1 + \sqrt{1+\alpha} b_2^+ a_2
\end{aligned} \tag{13}$$

4. CONCLUSION

We have obtained one-parameter homogeneous differential realization and the corresponding boson–fermion realization of the SPL(2,1) superalgebra. In terms of the conclusion it may be of use for further researches on one-parameter inhomogeneous differential realization and indecomposable and irreducible representations of the SPL(2,1) superalgebra.

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